
Chapter 7

The Rate of Return of Pay-As-You-Go Pension Systems: A More Exact Consumption-Loan Model of Interest

*Ole Settergren and Boguslaw D. Mikula**

Paul Samuelson's well-known article, "An Exact Consumption-Loan Model of Interest With or Without the Social Contrivance of Money," published in 1958, has been interpreted as showing that the rate of return of pay-as-you-go (PAYG) pension systems—that is, unfunded pension schemes—is the growth in the contribution, or tax, base of the system. In the absence of technological progress and with a constant number of hours worked per person, the growth in the contribution base is equal to the growth in population, or what Samuelson calls the "biological interest rate."

Several researchers have pointed out that the two-age overlapping generation (OLG) model used by Samuelson cannot explain the dynamics of the equilibrium interest rate in a world of more than two-age OLGs. Arthur and McNicoll (1978) and Willis (1988) have demonstrated that in a more than two-age OLG model, changes in the differential between the ages at which the average income is earned and consumed is a critical factor in determining equilibrium interest rates. Likewise, Keyfitz (1985, 1988), as well as Lee in numerous works (1980, 1988a, 1988b, 1994a, 1994b, 2000), showed that the amount consumed at some or all ages is affected by changes in this age differential. However, statements that the rate of return on PAYG financing is equal to the growth in the contribution base are surprisingly common.¹ Rarely are such claims accompanied by the necessary qualification that they are valid only in a two-age OLG model or in the equally unrealistic case of an economy and demography characterized by a steady state.²

The common assumption that the rate of return of PAYG pension systems is equal to the growth in the contribution base is rarely an efficient simplification. Recent experience in

* Ole Settergren is the director of the Pensions Department at the Swedish Social Insurance Agency and Boguslaw D. Mikula is a researcher in the Division for Analysis at the same agency.

We are grateful to Sergio Nisticò and Edward Palmer for their valuable comments. Special recognition goes to Hans Olsson. Without his early support, experience, and hard work preceding the decision by parliament in 1998, when it settled for an asymmetric design of the indexation of the new system's pension benefits and implicit pension liability, the ideas presented here would probably not have survived their infancy and become Swedish legislation.

With the permission of the World Bank, this chapter was published in the *Journal of Pension Economics and Finance*, volume 4, issue 2, July 2005.

Sweden indicates how poor that assumption can be. The increase in life expectancy in Sweden from 1980 to 2001 made the income-weighted average age of retirees increase from 72 to 75 years, while the income-weighted average age of contributors to the system remained relatively stable at 43 years.³ Thus the increase in life expectancy made the differential between the average age at which contribution was paid to the system and the average age at which pension was paid from it grow from about 29 to 32 years. This represents close to a 12 percent increase, which added roughly 0.4 percentage units to the annual real rate of return on contributions to the PAYG system during this period. As the average annual growth in the contribution base over the same period was only 0.3 percent, the common simplification that the rate of return equals the growth in the contribution base revealed less than half of the true return.

Thus one consequence—and perhaps a counterintuitive one—of increases in life expectancy is that it adds to the rate of return of PAYG pension systems. This indicates an even more serious drawback with the simplified view than its low precision. The assumption that the rate of return of PAYG financing is the growth rate of the contribution base hides a structure vital to understanding the financial dynamic of PAYG pension systems. Possibly the simplification is so frequent because it is assumed that, without it, the analysis becomes prohibitively complex for making statements about the system's cross-section internal rate of return (IRR).

The aim of this study is to demonstrate that there is a simple method for estimating the cross-section internal rate of return on contributions to PAYG pensions systems, even as the two-age OLG restriction, and the steady state restriction, is canceled. The method entails a procedure for valuing the contribution flow of PAYG financing and identifies the complete set of factors that decides the cross-section IRR. The procedure makes it possible to apply the algorithm of double-entry bookkeeping in PAYG pension systems.

The method presented here for calculating the rate of return of PAYG pension financing, including the valuation of the contribution flow and the use of double-entry bookkeeping in a PAYG context, is a result of the research undertaken to deal with some conflicting ambitions of the new Swedish pension system.⁴ This method for solving, or rather managing, this conflict was reached in ignorance of the research cited above by Arthur and McNicoll, Willis, Keyfitz, and Lee.⁵

In this text, the phrase *cross-section* IRR is used to indicate a measure distinct from the more familiar *longitudinal* IRR measure, which informs the rate of return that equates the value of the time-specific contributions with the benefits to an individual or a group of individuals. The cross-section IRR is the return on the pension system's liabilities that keeps the net present value of the pension system unaltered during a period of arbitrary length. However, to derive the cross-section IRR, a continuous time model is used. The expression *cross-section internal rate of return* is abbreviated below to *rate of return*, while we sometimes use the abbreviation IRR. We also use the terms *contribution base*, *contribution rate*, and *contributions*, where some would prefer *tax base*, *tax rate*, and *taxes*.

In the next section, a method for estimating the value of the contribution flow to PAYG pension systems is presented. In the following section, this method is used to obtain a formula for calculating the cross-section IRR on contributions to such systems. The final section concludes by commenting on the results. In annex B, the methods used to value the contribution flow and the definition of the cross-section IRR are illustrated by means of simple numerical examples. Some readers will probably find it helpful to read the numerical examples before they read the second and third sections.

The Value of the Contribution Flow

PAYG financing implies that the flow of future contributions is used to finance a pension liability that has already accrued.⁶ It is probably a matter of personal preference whether one considers that a PAYG system, by definition, has a deficit equal to this liability, or accepts that its net present value is zero if contributions match pension payments. Here the latter view is taken and *financial balance* is defined as

$$\text{Assets} - \text{Liabilities} = 0. \quad (7.1a)$$

This standard definition of financial balance is unconventional for PAYG pension systems. The commonly performed projections of cash flows to and from PAYG pension systems used to evaluate their financial situation have traditionally not been presented in the form of assets and liabilities, as the methods used does not allow this.⁷ As already indicated, it seems reasonable to consider that a PAYG pension system whose contributions and benefits match have a zero net present value and subsequently to conclude that its liability is matched by an implicit asset, below referred to as a *contribution asset*. In another context, Lee (1994 and later) calls a corresponding concept *transfer wealth*.

Often PAYG systems are considered to be defined by the absence of any funded assets. In practice, however, there is normally a transaction account, and sometimes there are substantial funded assets. Systems without any funded assets are only a special case of the general description that follows. Hence equation 7.1a can be re-expressed as

$$CA(t) + F(t) - PL(t) = 0, \quad (7.1b)$$

where

- CA = contribution asset,
- F = buffer fund,
- PL = pension liability, and
- t = time.

In a steady state, contributions will equal pension benefits, thus $CA(t_{ss}) = PL(t_{ss})$, and $F(t_{ss}) = 0$. For each income and mortality pattern and set of pension-system rules, there is a unique value for the pension liability. Equations 7.2 through 7.4 give an expression for this value in steady state.

In the case of a stable population—that is, a population with constant mortality rates and constant population growth—the age distribution of the population can be expressed as

$$N(x) = N(0) \cdot l(x) \cdot e^{-\gamma x}, \quad (7.2)$$

where

- $N(x)$ = number of persons of age x ,
- x = age,
- γ = the rate of fertility-driven⁸ population growth, and
- $l(x)$ = life table survival function.

In this system, the indexation of benefits can have any relation to the growth in average wage. Thus the pension benefit may vary in size relative to this average wage at different

ages. If, for example, pensions are indexed by the change in consumer prices, and average wages grow at a faster rate, the average pension benefit per birth cohort will be lower for older cohorts than for younger ones. The distribution of pensions within a cohort is ignored, since it has no relevance for the system-level rate of return.

The pension liability, V , is defined as the present value of future pension benefits to all persons to whom the system has a liability at the time of evaluation minus the present value of future contributions by the same individuals,

$$V = \int_0^m \text{population}(x) \int_x^m PV[\text{pensions}(u) - \text{contributions}(u)] du dx, \quad (7.3)$$

where

m = maximum age, and

x, u = age; both are variables of integration.

Discounting payments to and from the pension scheme by the growth in the contribution base, the pension liability can be re-expressed as:

$$V = \int_0^m \underbrace{N(0) \cdot l(x) \cdot e^{-\gamma \cdot x}}_{\text{population, age } x} \int_x^m \underbrace{\frac{l(u)}{l(x)}}_{\text{survivor rate}} \cdot \underbrace{e^{-\gamma \cdot (u-x)}}_{\text{discounting}} \cdot \left[\underbrace{k \cdot \bar{W} \cdot e^{\phi \cdot u}}_{\text{pension payments}} \cdot R(u) - \underbrace{c \cdot \bar{W} \cdot W(u)}_{\text{contributions}} \right] du dx, \quad (7.4)$$

where

$W(x)$ = wage pattern—that is, the average wage for age group x , as a ratio of the average wage for all age groups,

\bar{W} = average wage in monetary units per unit of time,

c = required contribution for a financially stable PAYG pension system,

ϕ = the rate of pension indexation relative to the rate of average wage growth,

$R(x)$ = number of retirees as a ratio of the number of individuals in age group x ,
and

k = constant determining the pension level (equals the replacement rate if $\phi = 0$).

The rate of discount is the product of the growth in average wages times the rate of population growth. As both wages and benefits grow with the average wage growth, the growth in average wage falls out of the equation and leaves the population growth rate as the effective discount rate, γ . It would be inappropriate to use a market rate of return on capital as a discount rate. The return on capital has no impact on the financial balance of a PAYG pension system, disregarding its effect on the buffer fund if there is one.

For a stable population with stable income patterns, the contributions, C , are generated by the size of the population by age, $N(x)$; the wage pattern, $W(x)$; the average wage, \bar{W} ; and the required contribution rate for a financially stable system, c .

$$C = \int_0^m N(x) \cdot c \cdot \bar{W} \cdot W(x) dx. \quad (7.5)$$

In a steady state, the contribution rate that satisfies the financial-stability criteria of equation 7.1 is also the contribution rate that makes contributions equate with pension payments in every period. Thus c is calculated as

$$\underbrace{\int_0^m N(0) \cdot l(x) \cdot e^{-\gamma x} \cdot k \cdot \bar{W} \cdot e^{\phi x} \cdot R(x) dx}_{\text{pension payments}} = \underbrace{\int_0^m N(0) \cdot l(x) \cdot e^{-\gamma x} \cdot c \cdot \bar{W} \cdot W(x) dx}_{\text{contributions}},$$

$$c = k \cdot \frac{\int_0^m e^{-(\gamma-\phi)x} \cdot l(x) \cdot R(x) dx}{\int_0^m e^{-\gamma x} \cdot l(x) \cdot W(x) dx}. \tag{7.6}$$

It is possible to get a form of measurement of the pension liability, in steady state, that is independent both of the size of the contribution base and of the contribution rate simply by dividing the pension liability by contributions paid per time unit. Thus equation 7.4 is divided by equation 7.5, where equation 7.6 is substituted for c . Rearranging and integrating by parts, this simplifies to

$$\frac{V}{C} = \frac{\int_0^m x \cdot [e^{-(\gamma-\phi)x} \cdot l(x) \cdot R(x)] dx}{\underbrace{\int_0^m [e^{-(\gamma-\phi)x} \cdot l(x) \cdot R(x)] dx}_{\text{average age of retirees}}} - \frac{\int_0^m x \cdot [e^{-\gamma x} \cdot l(x) \cdot W(x)] dx}{\underbrace{\int_0^m [e^{-\gamma x} \cdot l(x) \cdot W(x)] dx}_{\text{average age of contributors}}}. \tag{7.7}$$

The intermediate steps of the simplification are presented in annex C.

Equation 7.7 informs the conceivably intuitively reasonable fact that in steady state, the liability divided by contributions is equal to the time difference between the average age of retirees (the first term of the right-hand side), and the average age of contributors (the second term of the right-hand side). Both ages are money-weighted. However, this is not evident from the expression, as the average wage is a part of contributions, C . This leaves equation 7.7 with only the age patterns. The age difference between the average contributor and retiree is a measure of the duration of the pension liability. We call it *turnover duration (TD)*.

$$\frac{V}{C} = A_r - A_c = TD \tag{7.8}$$

where

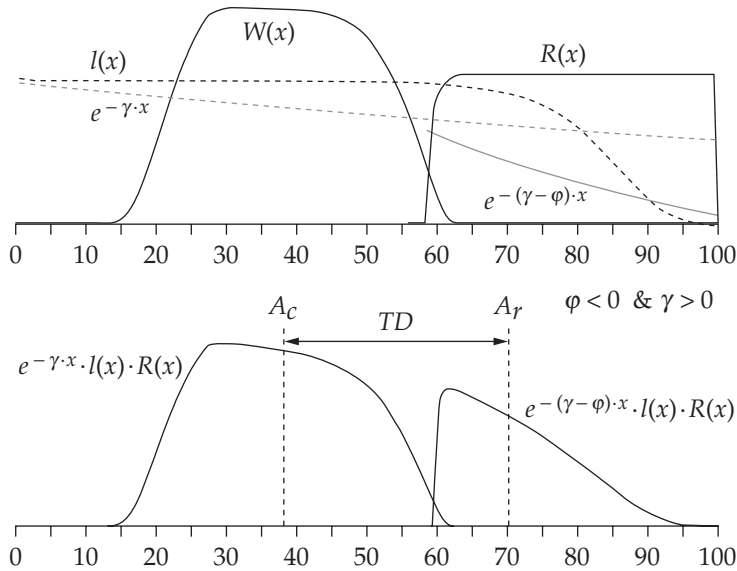
A_r = money-weighted average age of retiree, and

A_c = money-weighted average age of contributor.

The top half of figure 7.1 illustrates the age structure for average wage, $W(x)$; a certain life-table, $l(x)$; a population growth trend, γ ; and rules for indexation of pensions, ϕ . It also illustrates the retirement pattern, $R(x)$. The bottom half of figure 7.1 illustrates the resulting age structure for the contribution base and the pension payments. The resulting age-differential between the money-weighted average age of retirees, A_r and contributors, A_c is also shown.

Hence, for a stable population with stable income patterns, the factors determining the size of the pension liability can be separated into a volume component that is *contributions*

Figure 7.1. Illustration of Equations 7.7 and 7.8



Source: Authors.

and a structural component that is the *turnover duration*. Turnover duration is a useful concept that sums up the factors that determine the scaleless size of the pension liability: *scaleless* refers to disregarding the size of contribution rate and contribution base. The present value of the pension liability for a stable population with stable income patterns is expressed in years of contributions:

$$\frac{V}{C} = TD \Leftrightarrow V = TD \cdot C. \tag{7.9a}$$

The separation of the steady state pension liability into a volume and a structural component also has a temporal aspect. Except in a steady state, there will be no definite value for the turnover duration; however, the current economic and demographic patterns can be used to measure the *expected* turnover duration. It is expected in the same sense as the common measure of life expectancy; that is, it uses current observations to calculate a value that will turn out correct *ex post* only if observed patterns remain constant. The probability that any generation will live according to any published life table is virtually zero. Nevertheless, life tables are relevant and useful. Repeated estimations of the expected turnover duration⁹ will reflect the changes in the financially relevant patterns and thus yield new estimates of the contribution asset, which are infinitely unlikely to produce the *ex post* correct figure. This procedure of repetitive revaluation of the contribution asset is not so different from the recurring revaluation of funded assets by the market.¹⁰ For these reasons we find it appropriate to define the value of the contribution flow as the current turnover duration times the current contributions:

$$CA(t) = TD(t) \cdot C(t). \tag{7.9b}$$

The turnover duration indicates the size of the pension liability that the present contribution flow can finance, given the present income and mortality patterns and the population growth rate. As economic and demographic patterns change, the new value of the contribution flow can be estimated. The inverse of the turnover duration is a computable discount rate for the contribution flow, a measure of the current internal time preference of the PAYG pension scheme. This time preference is a function of the system design with respect to the rules that govern the indexing of pensions, the income and mortality patterns of the insured population, and the population growth trend. Annex C provides rough estimates of the turnover duration for 41 countries from Settergren and Mikula (2001). The country-specific turnover duration varies from 31 to 35 years; thus, with the internal time preferences of the hypothetical pension system in the estimate, the discount rates for contributions vary between approximately 2.8 and 3.2 percent. These rates are interestingly close to the frequently assumed real interest rate of about 3 percent.

The usefulness of the turnover duration for valuing the contribution flow is critically dependent on its volatility. In many countries, perhaps most, the volatility of the turnover duration can be anticipated to be moderate to low. The stem-and-leaf exhibit in figure 7.2 presents estimates of the annual percentage change in the turnover duration in Sweden in the period 1981–2003.¹¹ The average increase was 0.4 percent, most of it attributable to the increase in life expectancy; the average, money-weighted age of contributors remained closely around age 43, with no clear trend. The maximum one-year increase in turnover duration was 2.1 percent; the maximum annual decrease was 0.5 percent. More than half, 12, of the annual changes were between zero and 0.5 percent, and the standard deviation of the 23 observations was 0.6.

In the next section, the above method for estimating the value of the contribution flow is used to derive an expression for the rate of return of PAYG pension systems, and the application of double-entry bookkeeping for such systems is outlined.

Figure 7.2. Turnover Duration in Sweden, 23 Annual Changes, percentages 1981–2003

2	1
1	8
1	11
0	865
0	4443220000
-0	234
-0	5

The stem-and-leaf diagram is read as follows:

...	...
1	11 = 1.1% and 1.1%
0	865 = 0.8%, 0.6% and 0.5%
etc.	

Source: Riksförsäkringsverket (2003, 2004).

The PAYG System's Rate of Return

Financial balance can be ensured by adjusting the size of the pension liability: that is, by adjusting the value of present and or future benefits, or by adjusting the contribution rate—the size of the contribution asset as defined by equation 7.9b—or by doing both. Irrespective of the type of tuning, the financial-balance requirement of equation 7.1b, a net present value of zero, applies. To continue the derivation of the IRR, equation 7.1b is rephrased to

$$TD \cdot C + F - PL = 0, \quad (7.10)$$

where

F = buffer fund, and

PL = pension liability.

Equation 7.10 implies that both negative and positive funded assets are allowed, and in some situations necessary, to comply with the definition of financial balance.¹² Annex B presents numerical examples that illustrate this point.

The rate of return of the pension liability that yields a net present value of zero is, by definition, the rate of return on contributions to the system. The formula for the rate of return of a PAYG pension system follows from differentiating equation 7.10 with respect to time

$$\frac{d(TD \cdot C + F - PL)}{dt} = TD \cdot \frac{dC}{dt} + \frac{dT D}{dt} \cdot C + \frac{dF}{dt} - \frac{dPL}{dt} = 0. \quad (7.11)$$

The change in pension liability is a function of the rate of return of the liability and of the difference between payment of contributions and disbursements of pensions, as shown in equation 7.12

$$\frac{dPL}{dt} = PL \cdot IRR + (C - P), \quad (7.12)$$

where

IRR = internal rate of return, and

P = pension payments in monetary units per unit of time.

The IRR may turn up in two different forms: *implicit* and *explicit*. The *implicit* rate of return is a function of the impact of changes in mortality on the pension liability, and of any divergence between new pension obligations and contributions paid. In addition, changes in the rules of the system will normally alter the value of the pension liability, producing an implicit effect on the IRR. The *explicit* rate of return is the result of any explicit rules for indexing the liability—that is, the benefits to present and future retirees.

The net difference in payments to and from the pension system is captured by the buffer fund, if there is one. In addition, the value of the fund is changed by the return on its assets, as shown in equation 7.13

$$\frac{dF}{dt} = F \cdot r + (C - P), \quad (7.13)$$

where

r = rate of return on the buffer fund.

Depending on its sign and magnitude, the return on the buffer fund may increase or decrease the rate of return of the PAYG pension system. Equation 7.11 can then be re-expressed as

$$TD \cdot \frac{dC}{dt} + \frac{dT D}{dt} \cdot C + F \cdot r - PL \cdot IRR = 0 . \tag{7.14}$$

Finally, the IRR, separated into its components, is

$$IRR = \underbrace{\frac{TD \cdot dC}{PL \cdot dt}}_i + \underbrace{\frac{dT D}{PL \cdot dt} \cdot C}_{ii} + \underbrace{\frac{F \cdot r}{PL}}_{iii} . \tag{7.15}$$

Thus the rate of return of a PAYG system is a function of:

- i** *Changes in contributions* This component consists of Samuelson’s biological interest rate, changes in labor force participation, average wage growth, and changes in the contribution rate.
- ii** *Changes in turnover duration* This component consists of changes in income and mortality patterns and in the fertility-driven growth rate of the population.¹³
- iii** *Buffer fund return (interest)* This component consists of the return (interest) on any assets (debt) in the system.

The part of the IRR that is caused by mortality changes and any divergence between new pension obligations and contributions paid, or by changes in the system rules, can be considered an implicit indexation of the pension liability. The IRR reduced by the rate of implicit indexation is the rate of available indexation of the pension liability. Thus,

$$\text{Rate of available indexation} = i + ii + iii - \text{rate of implicit indexation} . \tag{7.16}$$

In practice, the indexation rules, or the adjustment of the contribution rate or other system rules, are not necessarily such that they distribute all the available indexation each time period. The applied indexation will differ temporally from what is available. The difference is the net income or loss to the system during the measured time period.

$$\text{Rate of available indexation} - \text{rate of explicit indexation} = \text{system net income} . \tag{7.17}$$

The accrued value of such net income or losses gives the opening surplus or deficit for the next period.

How does the expression for the cross-section IRR of PAYG pension systems in equation 7.15 relate to the longitudinal IRR on contributions? For individuals, this calculation becomes possible at the time of death. For a birth cohort, it is possible when everyone in

the cohort has died; for the pensions system, it is possible when it has been closed down. Such delays of information are, indeed, impractical. Both participants and policy makers want regular information on financial position and development. In order to produce this information, cross-section rate of return measures must be adopted, for which there is only imperfect information. In the business world, the problem is similar. The true rate of return can be calculated only when all payments to and from the business entity have been made. As business stakeholders need regular information on the rate of return, accounting principles are developed for an ongoing business. It is well known that accounting measures of the rate of return—basically business net income—are subject to some degree of arbitrariness. Therefore, the preferred choice of method is a matter for debate.

For PAYG pension systems, it is possible to envisage other accounting procedures than the one described here, and other measures will normally yield a different rate of return for a specific period. By our method, the contribution flow is valued according to the turnover duration with cross-section observations at the time, while the pension liability is estimated with an actuarial projection that may or may not imply changes in future turnover duration. Such differences will have an impact on the trajectory of the measured rate of return, but not on the aggregate rate of return as the system approaches a hypothetical steady state.

Conclusions

The rate of return on contributions to PAYG pension systems is not only a function of the growth in the contribution base of the system. It is also a function of changes in income and mortality patterns and in the trend of population growth. These three factors cause changes in the average age at which contributions are paid and pensions received: that is, they cause changes in *turnover duration*. Further, if there is a buffer fund in the system, the return on that fund will influence the rate of return on contributions. The rate of return can be implicitly distributed through the effects on the pension liability from mortality changes and also by differences between contributions paid and new pension liabilities. The difference between the rate of return and the implicitly distributed return is the rate of available indexation—the explicit indexation of the pension liability, which must be made to keep the systems net present value unaltered.

The turnover duration provides an estimate of the discount rate for the contribution flow to systems, which finances obligations with a zero prefunding requirement, that is, PAYG systems. This makes it possible to apply a form of double-entry bookkeeping. With the double-entry algorithm, the financial position of these schemes can be reported by means of a balance sheet, as summarized in equation 7.10, and changes in the financial position can be reported by means of an income statement, summarized in equation 7.17.¹⁴ We would argue that extending the field of double-entry bookkeeping to PAYG pensions systems has the potential of improving the quality and transparency, and thus the understandability, of financial information on these important transaction systems over the different measures of actuarial balance used today. Disentangling the components of the rate of return also adds options for the design of PAYG pension systems. In particular, the forms of indexing pensions can be given a more efficient design.¹⁵

Annex 7A. A Bibliography of the Legislative History of the Income Index and the Automatic Balance Mechanism

- Settergren, O. With the assistance of H. Olsson. 1997. Kapitel 5 Inkomstindex "Inkomstgrundad ålderspension—finansiella frågor m.m.," Ds 1997:67. Socialdepartementet, Stockholm. (Chapter 5 The Income Index in "Income-Related Old-Age Pension—Financial Issues etc." Ds 1997:67, Ministry of Health and Social Affairs, Stockholm.)
- . 1998. Chapter 16 Inkomstindex, "Regeringens proposition 1997/98:151 Inkomstgrundad ålderspension, m.m.," Riksdagen, Stockholm. (Chapter 16 The Income Index in "The Government Bill 1997/98:151 Income-Related Old-Age Pension etc." Swedish Parliament, Stockholm.)
- . 1999. With the assistance of H. Olsson and D. Sundén. "Automatisk balansering av ålderspensionssystemet—regler för avsteg från inkomstindexeringen inom ålderspensionssystemet," Ds 1999:43. Socialdepartementet, Stockholm. (Automatic Balancing of the Old-Age Pension System—Rules for Deviating from the Income Indexation within the Old-Age Pension System." Ds 1999:43, Ministry of Health and Social Affairs, Stockholm.)
- . 2000. With actuarial modeling and calculations by B. D. Mikula, N. Holmgren, and J. Leander. "Automatisk balansering av ålderspensionssystemet—redovisning av regeringens beräkningsuppdrag," RFV Analyserar 2000:1 Riksförsäkringsverket, Stockholm ("Automatic Balancing of the Old-Age Pension System—An Account of the Calculation Assignment from the Government." RFV Analysis 2000:1).
- . 2001. "Regeringens proposition 2000/01:70 Automatisk balansering av ålderspensionssystemet," Riksdagen, Stockholm. (The Government Bill 2000/01:70 Automatic Balancing of the Old-Age Pension System." Swedish Parliament, Stockholm.)

Annex 7B. Numerical Illustrations in an Overlapping Generation Model

A three-age overlapping generation (OLG) model is used to illustrate the impact of changes in the average ages at which income is earned and consumed. Three is the minimum number of ages needed for changing the differential between the ages at which the average income is earned and consumed. This age differential is called *turnover duration*, and is formally derived in equations 7.1–7.9. To demonstrate the effects of changes in mortality on the rate of return, the model is extended from three to four ages.

In the model, the life of an individual is divided into three (four) periods of equal length. All individuals work for exactly two periods, at ages 1 and 2, and they are all retired for the entire third (and fourth) period, age 3 (and 4). All are born on the first day of each period; all birth cohorts are of equal size; there is no fertility-driven population growth, no migration, and no preretirement mortality. And everyone in retirement dies on the last day of their final period. There is no technological progress. Under these assumptions, the contribution base for the pension system is constant. All financial transactions are made at the end of each period. To avoid the complication that changes in contribution rate have on the internal rate of return (IRR) (see equation 7.15) the examples are constructed so that the system in all examples can finance its pension payments with an unchanged contribution rate—25 percent—for every period in all examples.

The effects on IRR from shifts in income and mortality patterns are described for certain alternative pension-system rules. The reason for this is to illustrate that:

- The system's cross-section IRR is not a function of system design¹
- The distribution of the IRR over cohorts, the "longitudinal IRR,"² is a function of system design
- The timing of cash flows is a function of system design, even when designs are equally financially stable in the sense that they all produce a zero net present value as defined in equation 7.10.

Although the numerical examples are straightforward, the somewhat complex feature of OLG, in combination with the detailed account of the effects of the shifts in income and mortality patterns, may make it tedious to work through the examples. However, this effort can be well invested as the examples, once grasped, clearly reveal structures that are vital for understanding important aspects of PAYG financing.

Example 1. A Shift in Income Pattern

Summary of what the example illustrates. In the following example, the income pattern shifts—the income of older workers increases relative to that of younger workers, so that the income-weighted average age of contributors increases. The example shows how this change decreases the turnover duration and leads to a negative IRR. The effects of the negative IRR are illustrated for a pension system where the rules are such that this specific shock will result in a *rate of implicit indexation* equal to the negative IRR. In example 1.1 below, the effects of the same shift in income pattern are illustrated for a system where the rules are such that this specific shock will result in a rate of implicit indexation equal to zero. Thus in example 1.1, to maintain a zero net present value, the negative IRR must be distributed through *explicit* indexation equal to the IRR. The subsequent effects on the cash flow, buffer fund, and so on of the system are illustrated with an income statement and a balance sheet.

The shift in income pattern. Up until and including period 1, the wage is 48 for the older working cohort and also 48 for the younger. In the period 2, the income pattern is

changed.³ From then on, the wage is 72 for the older cohort and 24 for the younger. Thus the wage sum, equal to the contribution base, is constantly 96. The average wage for workers in general also remains unaltered; only the distribution of the average wage between the age groups has shifted.

The rules of the pension system. The pension system is designed to pay a benefit that is 50 percent of the gross average wage of all wage earners—admittedly an awkward rule, but here it serves our purpose.

The effects of the shift in income pattern. Table 7B.1 illustrates that this system will result in contributions of 24 that perfectly match pension benefits of 24 before and after the shift in income pattern. Cohort B, the only cohort whose lifetime income is altered by the change in income pattern, will receive a pension of 24, whereas it paid contributions of 30, the sum of 25 percent of wages 48 and 72, respectively. As the pension received is only 24, this cohort will receive 6 less than they paid, a periodically compounded rate of return of roughly -15 percent. The computation is

$$0.25 \times 48 \times r^2 + 0.25 \times 72 \times r = 24 \Rightarrow r - 1 \approx -15\%.$$

The effect of the change in income pattern on the system’s cross-section rate of return is the monetary effect, -6, relative to the systems pension liability⁴ of 36. Thus the cross-section rate of return is -1/6. Table 7B.1 shows that the cross-section rate of return is equal to the relative decrease in the money-weighted average difference in time between the payment of contributions and the collection of benefits, that is, the decrease in turnover duration from 1.5 to 1.25. From table 7B.2 it is also clear that the change in turnover duration makes the con-

Table 7B.1. Effect of a Shift in Income Pattern on Cohort Contributions and Benefits

Cohort \ Period	0	1	2	3	4	Cohort total	
						Contributions	Pensions
A	48	48	24			24	24
B		48	72	24		30	24
C			24	72	24	24	24
D				24	72	24	..
Wage sum	..	96	96	96	..		
Period total	Contribution rate	25%	25%	25%	25%	25%	
	Contributions	..	24	24	24	..	
	Pensions	24	24	24	

Source: Authors.

Note: The central box shows wage sums in normal type and **pensions in bold face**, per period, for each cohort.

Table 7B.2. Effect of a Shift in Income Pattern on Turnover Duration and Pension Liability

	<i>Before shift</i>	<i>After shift</i>	<i>Relative change</i>
Average age of retiree, \bar{A}_R	3 ^a	3 ^a	—
Average age of contributor, \bar{A}_C	1.5 ^b	1.75 ^c	+1/6
Turnover duration, $TD, (\bar{A}_R - \bar{A}_C)$	1.5	1.25	-1/6
Contribution asset, $TD \times$ contributions	36	30	-1/6
Pension liability, PL	36 ^d	30 ^e	-1/6
IRR (Monetary loss / PL) ^f		-6/36	-1/6

Source: Authors.

Note: In explanations b, c, d, and e, contributions are shown in normal type, **pensions in bold face**, ages in italics. For explanation of pension liability, brackets [] are used to group money flows from and to the same cohort. Figures relating to cohorts are presented in order from oldest to youngest.

a. All pensions are paid at age 3.

b. $(48 \times 2 + 48 \times 1) / (48 + 48)$.

c. $(72 \times 2 + 24 \times 1) / (72 + 24)$.

d. $[24] + [24 - 12]$.

e. $[24] + [24 - 18]$.

f. The monetary loss occurs "at" the shift, indicated by the placement of the number between the "before shift" and "after shift."

tribution asset, which is calculated as the contribution flow times the turnover duration, decrease and that this decrease is equal to the monetary loss incurred by cohort B.

As a combined effect of the shift in income pattern and the rules of this pension system, the pension liability decreases to the same extent that the value of the contribution flow is reduced by the shorter turnover duration. Before the shift the pension liability was 36; after the shift the pension liability is 30. Owing to this implicit negative indexation of the pension liability, the net present value of the system is consistently zero throughout the shift. The shift in income pattern in combination with the rule that says that pensions are 50 percent of average income of all wage earners implicitly distributes the negative IRR to cohort B. However, the negative IRR itself was not a consequence of the system's rules, as will be illustrated in the following example.

Example 1.1 Same Shift in Income Pattern, Different Pension System Rules

The rules of the pension system. The same shift in income pattern is now applied in a pension system designed as a so-called notional defined contribution (NDC) system. The rules of such a system imply that each cohort is to be repaid an amount equal to their contributions indexed at some rate, positive or negative. Initially, the indexing rules of the system are assumed to provide that notional pension capital and pensions are to be revalued at the growth rate of the contribution base, which in the example is zero for every period.

The effects of the shift in income pattern. Up until and including cohort A and period 2, this system will yield an identical result as the first set of rules: zero cross-section and longitudinal IRRs. But when cohort B retires, it will have accumulated a notional pension capital of 30, equal to what it has paid in contributions. As the flow of contributions is constant at 24, the system can only repay cohort B their notional pension capital by incurring a deficit of 6—a figure familiar from example 1. This deficit is caused by the same reduction in turnover duration as in example 1. However, in the NDC system the same negative IRR causes a cash deficit since the rate of (implicit) indexation is zero, while in Example 1 the rate was -1/6, thus distributing the negative IRR.

The shift in income pattern does not immediately reduce the pension liability of the notionally defined contribution system; this liability remains at 36 in period 2,⁵ while the shorter turnover duration—just as in Example 1—has decreased the value of the contribution flow to 30. To be financially stable, the notional defined contribution pension system must explicitly distribute the negative IRR by reducing the pension liability. One way to accomplish this is to index the system's total pension liability by the "rate of available indexation," (see equation 7.16). Table 7B.3 shows the development of the income statement and balance sheet of the NDC system, which applies explicit indexation at the available rate, here equal to the IRR.

Indexing cohort B's and C's notional pension capital of 30 and 6,⁶ respectively, by the available rate of 5/6 reduces it to 25 and 5, respectively. Thus the total pension liability of

Table 7B.3. Example 1: Income Statement, and Balance Sheet

	<i>Period</i>				
	1	2 ^a	2	3	4
<i>Income statement</i>					
Contributions	24	24	24	24	24
Pensions	-24	-24	-24	-25	-23
= net cash flow (a)	0	0	0	-1	1
Change in contribution asset (b)	0	-6 ^c	-6 ^c	0	0
New accrued pension liability ^b	-24	-24	-24	-24	24
Paid-off pension liability ^b (= paid pension benefits)	24	24	24	25	23
Indexation of liability ^b	0	0	6	0	0
= change in pension liability (c)	0	0	6	1	-1
Net income/ -loss, (a) + (b) + (c)	0	-6	0	0	0
<i>Balance sheet</i>					
Buffer fund	0	0	0	-1	0
Contribution asset	36	30	30	30	30
= total assets (d)	36	30	30	29	30
Pension liability, age 3	0	0	0	0	0
Pension liability, age 2	24	30	25 ^d	23 ^e	24
Pension liability, age 1	12	6	5 ^e	6	6
= total liability (e)	36	36	30	29	30
Net present value of system (d) - (e)	0	-6	0	0	0

Source: Authors.

a. Values before indexation with the available rate of return.

b. A negative number denotes an increase in the pension liability and thus a cost. A positive number denotes a decrease in the pension liability and thus an income.

c. $-0.25 \times 24 = -6$ [change in $TD \times [\text{contributions}(t) + \text{contributions}(t-1)]/2$].

d. $12 + 18 \times 5/6 = 25$ [cohort B's period 1 contribution + cohort B's period 2 contribution \times IRR].

e. $6 \times 5/6 + 18 = 23$ [cohort C's period 2 contribution \times IRR + cohort C's period 3 contribution].

the system is reduced from 36 to 30—equal to the new shorter turnover duration of the system (1.25) times the contribution flow (24). This implies that the system has regained its zero net present value. Nonetheless, the shift in income pattern and the negative indexation of the pension liability will affect the system's cash flows. Period 3 pension payments to cohort B will be 25. As system income is 24 every period this will result in a deficit of 1. In period 4, pension payments to cohort C will be 23 (5 + 18); thus a cash flow surplus of 1 will arise and close the deficit.⁷ The systems total assets period 3 are 29, that is the sum of the buffer fund period 3 is -1 and the contribution asset is 30. The total assets are equal to the system's pension liability, and the system's net present value is consistently zero.

Example 2. A Shift in Mortality Pattern

Summary of what the example illustrates. In example 2 the mortality pattern changes—life expectancy shifts upward—so that the money-weighted average age of retirees increases. The example shows how this change increases the turnover duration and results in a positive IRR. The effects of the positive IRR are illustrated for a pension system where the rules are such that the rate of implicit indexation equals the positive IRR. In example 2.1 the effects of the same shift in mortality pattern are illustrated for a system where the rules are such that the rate of implicit indexation is zero. Thus in example 2.1, to maintain a zero net present value, the positive IRR must be distributed through *explicit* indexation equal to the IRR. The subsequent effects on the cash flows, buffer fund, and so on of the system are illustrated with an income statement and a balance sheet.

The shift in mortality pattern. The shift in mortality occurs—simply, though unrealistically—through a one-time increase in life span. After one period of retirement, no retiree in cohort B dies; instead, after the third period, all retirees in the cohort continue to live for exactly one more period. Subsequent cohorts also live for exactly two periods as retirees.

The rules of the pension system. In the example, we keep the contribution rate fixed at 25 percent. Thus average pension benefit must be halved after the first cohort with a longer life expectancy received its first pension payment. Cohort B's pension is thus 24 in their first period as retirees and 12 in their second. Cohort C, the second cohort with a longer life span, will receive a pension of 12 in each period, as will subsequent cohorts.

The effects of the shift in mortality pattern. Table 7B.4 illustrate that the system will result in contributions of 24 that perfectly match pension benefits of 24 before and after the shift in mortality. However, cohort B, the first to benefit from the longer life span, will receive a total pension of 36, whereas it paid only 24 in contributions, for a periodically compounded rate of return of approximately 25 percent. The computation is

$$0.25 \times 48 \times r^3 + 0.25 \times 48 \times r^2 = 24 \times r + 12 \Rightarrow r - 1 \approx 25\%.$$

The effect of the change in mortality pattern on the system's cross-section rate of return is cohort B's monetary gain, 12, relative to the systems pension liability 36. Thus the cross-section rate of return is 1/3. Table 7B.5 shows that the cross-section rate of return is equal to the relative increase in the money-weighted average difference in time between the payment of contributions and the collection of benefits—that is, the increase in turnover duration from 1.5 to 2. The reason for the positive return is the longer time span between the average wage-weighted age of contributors and the average benefit-weighted age of retirees resulting from the shift in mortality pattern, the increase in turnover duration. With the longer turnover duration, the value of the contribution flow increases from 36 to 48.

The system is financially balanced throughout the shift since the pension liability increases to the same extent as the value of the contribution flow. The positive return of 12 is implicitly distributed to the cohort whose initial pension was calculated on the basis of

Table 7B.4. Effect of a Shift in Mortality on Cohort Benefits

Cohort \ Period	Period							Cohort total	
	-1	0	1	2	3	4	Contributions	Pensions	
A	48	48	24				24	24	
B		48	48	24	12		24	36	
C			48	48	12	12	24	24	
D				48	48	12	24	...	
Wage sum		..	96	96	96		
Period total	Contribution rate	25%	25%	25%	25%	25%	25%		
	Contributions	..	24	24	24		
	Pensions	24	24	24	24		

Source: Authors.

Note: The central box shows wage sums in normal type and **pensions in bold face**, per period, for each cohort.

Table 7B.5. Effect of a Shift in Mortality Pattern on Turnover Duration and Pension Liability

	Before shift	After shift	Relative change
Average age of retiree, \bar{A}_R	3 ^a	3.5 ^f	+ 1/6
Average age of contributor, \bar{A}_C	1.5 ^b	1.5 ^b	—
$\bar{A}_R - \bar{A}_C$, turnover duration, TD	1.5	2	+ 1/3
Contribution asset, $TD \times$ contributions	36	48	+ 1/3
Pension liability, PL	36 ^d	48 ^g	+ 1/3
IRR (monetary gain / PL)		12/36	+ 1/3

Source: Authors.

Note: See table 7B.2 for explanation of the use of normal type, **bold-face**, and *italics*.

a., b., d: see table 7B.2.

f. $(12 \times 4 + 12 \times 3) / (12 + 12)$.

g. $[12] + [12 + 12] + [12 + 12 - 12]$.

the previous life expectancy. This can also be illustrated by placing the numbers in the example into equation 7.16

$$[\text{rate of available indexation}] = [i] + [ii] + [iii] - [\text{rate of implicit indexation}].$$

$$0 = 0 + 1/3 + 0 - 1/3$$

The positive return resulting from an increase in life expectancy is due neither to the design of the pension system nor to the imperfect knowledge of life expectancy assumed

in the example. If cohort B's life span had been known *ex ante* and the benefit had been reduced to 12 already in cohort B's first period of retirement, there would have been a surplus of 12 in period 2. Equation 7.16 would then read as follows

$$\begin{aligned} \text{[rate of available indexation]} &= \text{[i]} + \text{[ii]} + \text{[iii]} - \text{[rate of implicit indexation]}. \\ 1/3 &= 0 + 1/3 + 0 - 0 \end{aligned}$$

If the available indexation is not used to increase the pension liability, the identity requirement for financial stability—a zero net present value—is not met since an undistributed surplus then arises. Example 2.1 illustrates the effects of one rule for distributing this surplus.

Example 2.1 Same Shift in Mortality Pattern, Different Pension System Rules

The rules of the pension system. We now again assume an NDC system. This NDC system indexes its liability by the available rate of return. In the example, this return will equal the IRR since we assume perfect information on life expectancy. In such a system and with this information, the surplus of 12, representing a rate of available indexation of 1/3, will be distributed through indexation of the pension liability in period 2.

The effects of the shift in mortality pattern. Before the shift, pension payments and contributions will both be stable at 24. Periods 3, 4, and 5, pension payments will be 16, 30, and 26, respectively. Pension payments will be back at 24 as from period 6. Periods 3 and 4, the buffer fund, will thus stand at 8 and 2, respectively, and be back at zero as from period 5. (Readers are encouraged to verify these calculations.) The positive fund is necessary to balance the pension liability, which will temporarily exceed the contribution asset by the same magnitude as the value of the fund. Assuming revaluation of the pension liability at the rate of available indexation and, more realistically, imperfect information on life expectancy, the flow of payments will be different. Still, the system will maintain a zero net present value at all times and in a steady state end up with a zero buffer fund.

Summary of What the Examples Show

For financially stable PAYG pension systems, the examples showed that the cross-section IRR, regardless of system design and ability to forecast life span, is affected identically by changes in income and mortality patterns. We also learned that the distribution of the IRR among cohorts depends on system design and ability to forecast mortality. Furthermore, the principle of double-entry bookkeeping in PAYG pension systems has been illustrated.⁸

Notes

1. However, if the rate of return on buffer fund assets or the interest rate on a buffer fund deficit differs from the growth rate in the contribution base, the design of the PAYG system as one with or without buffer fund will have a (small) impact on the system's cross-section IRR.

2. See the first section of this chapter for a definition of *cross-section* and *longitudinal* IRR.

3. *Income pattern* is defined as the ratio of the average wage for each age group to the average wage for all age groups. A stable income pattern exists when this ratio is constant over time for all age groups.

4. *Pension liability*, or *PL*, is defined in equation 7.3 as the present value of future pension benefits to all persons to whom the system has a liability at the time of evaluation, minus

Table 7B.6 Example 2.1: Income Statement and Balance Sheet

	Period					
	1	2 ^a	2	3	4	5
<i>Income statement</i>						
Contributions	24	24	24	24	24	24
Pensions	-24	-24	-24	-16 ^g	-30 ^h	-26 ⁱ
=net cash flow (a)	0	0	0	8	-6	-2
Change in contribution asset (b)	0	12 ^c	12 ^c	0	0	0
New accrued pension liability ^b	-24	-24	-24	-24	-24	-24
Paid-off pension liability ^b (= paid pension benefits)	24	24	24	16	30	26
Cost of / income from indexation of liability ^b	0	0	-12	0	0	0
= change in pension liability (c)	0	12	-12	-8	6	2
Net income /-loss (a) + (b) + (c)	0	12	0	0	0	0
<i>Balance sheet</i>						
Buffer fund	0	0	0	8	2	0
Contribution asset	36	48	48	48	48	48
= total assets (d)	36	48	48	56	50	48
Pension liability, age 3	0	0	0	16	14	12
Pension liability, age 2	24	24	32 ^d	28 ^f	24	24
Pension liability, age 1	12	12	16 ^e	12	12	12
= total liability (e)	36	36	48	56	50	48
Net present value of system (d) – (e)	0	12	0	0	0	0

Source: Authors.

Note: NDC system and indexing at the available rate, in the example equal to the IRR.

a., b., see table 7B.3.

c. $0.5 \times 24 = 12$ [change in $TD \times [\text{contributions}(t) + \text{contributions}(t-1)]/2$].

d. $(12 + 12) \times 4/3 = 32$ [(cohort B's period 1 contribution + cohort B's period 2 contribution) \times IRR].

e. $12 \times 4/3 = 16$ [cohort C's period 2 contribution \times IRR].

f. $(12 \times 4/3) + 12 = 28$ [(cohort C's period 2 contribution \times IRR) + cohort C's period 3 contribution].

g. $32 / 2 = 16$ (cohort B's notional pension capital period 2 / life expectancy).

h. $16 + (28 / 2) = 30$ (cohort B's pension period 4 + cohort C's pension period 4).

i. $14 + (24 / 2) = 26$ (cohort C's pension period 5 + cohort D's pension period 5).

the present value of future contributions by the same individuals. As there is neither population growth nor technological progress, the contribution base will be constant; thus, the discount rate will be zero.

5. Pension liability to cohort B is 30, and to cohort C it is 6. Only after cohort B has passed through the system will the total pension liability drop to the new sustainable level of 30—disregarding the deficit of 6 caused by the shift in income pattern.

6. The total wage of cohort C period 2 is 24; with the contribution rate of 25 percent this will make cohort C's contribution and notional capital equal 6 this period.

7. The return on the buffer fund assets is assumed to equal the growth in contribution base which is zero.

8. The accounting standard used is a simplified version of the format developed and used since 2001 for the Swedish public pension system.

Annex 7C. Intermediate Steps before Equation 7.7

$$\frac{V}{C} = \frac{N(0) \cdot \bar{W} \cdot k \cdot \int_0^m \int_x^m l(u) \cdot e^{-\gamma u} \cdot \left[R(u) \cdot e^{\phi u} - W(u) \cdot \frac{\int_0^m e^{-(\gamma-\phi)a} \cdot l(a) \cdot R(a) da}{\int_0^m e^{-\gamma a} \cdot l(a) \cdot W(a) da} \right] du dx}{N(0) \cdot \bar{W} \cdot k \cdot \int_0^m l(x) \cdot e^{-\gamma x} \cdot W(x) \cdot \frac{\int_0^m e^{-(\gamma-\phi)a} \cdot l(a) \cdot R(a) da}{\int_0^m e^{-\gamma a} \cdot l(a) \cdot W(a) da} dx} \dots$$

The expression above can easily be reduced by elementary algebraic manipulations. However, to make it simpler, the following substitutions will be useful:

$$\begin{cases} F_R(a) = e^{-(\gamma-\phi)a} \cdot l(a) \cdot R(a) \\ F_W(a) = e^{-\gamma a} \cdot l(a) \cdot W(a) \end{cases}$$

$$\begin{aligned} & \int_0^m \int_x^m l(u) \cdot e^{-\gamma u} \cdot \left[R(u) \cdot e^{\phi u} - W(u) \cdot \frac{\int_0^m F_R(a) da}{\int_0^m F_W(a) da} \right] du dx \\ \dots = & \frac{\int_0^m \int_x^m l(u) \cdot e^{-\gamma u} \cdot \left[R(u) \cdot e^{\phi u} - W(u) \cdot \frac{\int_0^m F_R(a) da}{\int_0^m F_W(a) da} \right] du dx}{\int_0^m F_W(x) \cdot \frac{\int_0^m F_R(a) da}{\int_0^m F_W(a) da} dx} \\ = & \frac{\int_0^m \int_x^m l(u) \cdot e^{-\gamma u} \cdot \left[R(u) \cdot e^{\phi u} \cdot \int_0^m F_W(a) da - W(u) \cdot \int_0^m F_R(a) da \right] du dx}{\int_0^m F_R(a) da \cdot \int_0^m F_W(x) dx} \\ = & \frac{\int_0^m \left[\int_0^m F_W(a) da \cdot \int_x^m l(u) \cdot e^{-\gamma u} \cdot R(u) \cdot e^{\phi u} du - \int_0^m F_R(a) da \cdot \int_x^m l(u) \cdot e^{-\gamma u} \cdot W(u) du \right] dx}{\int_0^m F_R(a) da \cdot \int_0^m F_W(x) dx} \\ = & \frac{\int_0^m \left[\int_0^m F_W(a) da \cdot \int_x^m F_R(u) du - \int_0^m F_R(a) da \cdot \int_x^m F_W(u) du \right] dx}{\int_0^m F_R(a) da \cdot \int_0^m F_W(x) dx} \\ = & \frac{\int_0^m F_W(a) da \cdot \int_0^m \left[\int_x^m F_R(u) du \right] dx - \int_0^m F_R(a) da \cdot \int_0^m \left[\int_x^m F_W(u) du \right] dx}{\int_0^m F_R(a) da \cdot \int_0^m F_W(x) dx} \\ = & \frac{\int_0^m \left[\int_x^m F_R(u) du \right] dx}{\int_0^m F_R(x) dx} - \frac{\int_0^m \left[\int_x^m F_W(u) du \right] dx}{\int_0^m F_W(x) dx} \dots \end{aligned}$$

For the next step, the following identity is needed:

$$\left. \begin{aligned} \int_0^m \int_x^m f(u) du dx &= \int_0^m x \cdot f(x) dx \\ \text{proof:} \\ \int_0^m \int_x^m f(u) du dx &= \left[x \cdot \int_x^m f(u) du \right]_0^m - \int_0^m x \cdot (-f(x)) dx \\ &= m \cdot \int_m^m f(u) du - 0 \cdot \int_0^m f(u) du + \int_0^m x \cdot f(x) dx = 0 + 0 + \int_0^m x \cdot f(x) dx \end{aligned} \right\} .$$

Thus

$$\dots = \frac{V}{C} = \frac{\int_0^m x \cdot F_R(x) dx}{\int_0^m F_R(x) dx} - \frac{\int_0^m x \cdot F_W(x) dx}{\int_0^m F_W(x) dx} = \frac{\int_0^m x \cdot e^{-(\gamma-\varphi)x} \cdot l(x) \cdot R(x) dx}{\int_0^m e^{-(\gamma-\varphi)x} \cdot l(x) \cdot R(x) dx} - \frac{\int_0^m x \cdot e^{-\gamma x} \cdot l(x) \cdot W(x) dx}{\int_0^m e^{-\gamma x} \cdot l(x) \cdot W(x) dx}$$

QED.

Annex 7D. Rough Estimate of Turnover Duration in 41 Countries

Individuals who are not in the labor force and are 55 years or older are assumed to receive benefits from the pension system that on average amount to 50 percent of the average wage. Pensions are assumed to be indexed by the growth in the average wage, thus $\phi = 0$.

<i>Country</i>	<i>Year</i>	<i>Estimated population growth, $\gamma - 1$ (percent)</i>	<i>Turnover duration (years)</i>
Tajikistan	1991	3.8	35.3
Argentina	1990	1.5	34.1
Spain	1990	0.2	34.1
New Zealand	1990	0.8	34.0
Australia	1994	0.3	34.0
Kyrgyz Republic	1995	2.8	33.7
Israel	1994	1.8	33.6
Portugal	1992	0.1	33.3
Canada	1992	-0.1	33.3
Chile	1997	1.5	33.2
Romania	1992	0.5	33.2
Italy	1994	-0.3	33.2
United States	1995	0.2	33.1
Austria	1996	-0.4	33.1
Belgium	1994	-0.1	33.1
Ireland	1990	0.9	33.0
France	1995	0.2	33.0
United Kingdom	1996	-0.1	32.9
Hungary	1996	-0.1	32.8
Greece	1995	-0.1	32.8
Kazakhstan	1996	1.4	32.7
Slovak Republic	1995	0.8	32.7
Denmark	1994	-0.4	32.7
Sweden	1996	-0.3	32.7
Netherlands	1995	-0.3	32.7
Latvia	1996	-0.1	32.6
Norway	1996	0.1	32.6
Armenia	1993	1.6	32.5
Czech Republic	1996	-0.1	32.1
Slovenia	1993	-0.2	32.0
Estonia	1996	0.0	31.9
Belarus	1996	0.1	31.9
Poland	1996	0.4	31.8
Russian Federation	1995	-0.1	31.7
Germany	1994	-0.8	31.7
Bulgaria	1993	-0.2	31.6
Japan	1990	-0.4	31.6
Finland	1996	-0.4	31.6
Korea, Rep. of	1991	0.9	31.5
Ukraine	1993	-0.1	31.3
Moldova	1994	0.8	31.2

Source: Authors' calculations, based on UN and ILO statistics. For details, see Settergren and Mikula (2001).

Notes

1. While many examples could be cited to illustrate this point, here are just two of them: "As Paul Samuelson showed 40 years ago, the real rate of return in a mature PAYG system is equal to the sum of the rate of growth in the labor force and the rate of growth in productivity" (Orszag and Stiglitz 1999, p. 15). "The rate of return in a notional system can only be the rate of growth of the tax base that results from rising real wages and increasing numbers of employees (Samuelson 1958)" (Feldstein 2002, p. 7).

2. In the context of pensions, a *steady state* is defined as a situation where the average wage at each age, relative to the average wage for all ages, is constant over time and where the number of retirees at each age, relative to the total number of retirees, is constant over time: that is, where mortality rates are constant. Thus the definition of steady state is consistent with population growth (or decline) if the change rate remains constant over time.

3. See Riksförsäkringsverket (2004). To be more precise, the average ages refer to the *expected* average ages. The expected ages will correspond with actual average ages only if fertility-driven population growth, income, and mortality patterns are stable: that is, if they are in a steady state.

4. See the legislative history of the indexation of the Swedish pension system and Settergren (2001, 2002).

5. This ignorance is clear from the legislative history of the Swedish pension reform (annex A), as well as Settergren (2001). It is evident that we were not alone in being unaware of the works, or of their implications, that "explore the interface of richer demographic models and the overlapping generation models of economists" (Lee, 1994a). An example is Valdés-Prieto (2000), who observed that changes in income and mortality patterns influence the financial balance of a "notional defined contribution" PAYG pension scheme. However, Valdés-Prieto does not explain his observations by the effects that changes in income and mortality patterns have on the money-weighted age difference between the average ages when income is earned and consumed.

6. Pension liability is defined as the present value of future benefits to all persons to whom the system has a liability at the time of evaluation, minus the present value of future contributions by the same individuals (equation 7.3), sometimes also referred to as the implicit pension liability (see also Iyer 1999). The practical problems of measuring pension liability in PAYG schemes are often substantial. Depending on the system design, and the quality and availability of data, the estimate of pension liability may indeed be so uncertain that it is practically useless. This chapter does not deal with these important practical obstacles for applying the method suggested for estimating IRR and applying double-entry bookkeeping.

7. An example of the conventional presentation of financial status in a PAYG pension system is the analysis in the Annual Report of the Board of Trustees in the Federal Old-age and Survivors Insurance and Disability Insurance Trust Funds (2003).

8. The expression could be extended also to incorporate the effects of migration on the expected contribution-weighted average age of contributors. See Settergren and Mikula (2001) for such an extended interpretation of γ .

9. Below we will not use the full expression *expected turnover duration* to indicate that the turnover duration is measured outside of a steady state, but will refer only to *turnover duration*.

10. An obvious difference between repetitive re-evaluation of the contribution assets and recurring re-evaluation of funded assets is that funded assets are tradable, which make their prices much less "implicit." However, their valuation is inevitably hypothetical to some degree as long as they are not sold off.

11. The data for the estimate are the annual individual earnings and benefit records for all persons covered by the national pension scheme in Sweden from 1981 through 2003.

12. To get a zero buffer fund in steady state, either the steady state rate of return on the buffer fund or the interest rate paid on a deficit must equal the growth in the contribution base or the valuation of the fund must reflect an assumption of a return on capital different from the growth in the contribution base.

13. Note that since the turnover duration is affected, however mildly, by changes in the fertility-driven growth rate, γ , the IRR may differ from the contribution base growth even in the unrealistic case of constant mortality and income patterns. This points out the shortcomings in the two-age OLG model: it cannot represent the relevant geometry of the problem.

14. In practical applications equation 7.17 should be extended to accommodate the possibility of an opening surplus or deficit—that is, a difference between assets (buffer fund assets and contribution asset) and liabilities.

15. Whether or not the claim that double-entry accounting provides better information than traditional measures of actuarial balance is correct can, perhaps, be judged from the Annual Reports of the Swedish Pension System, which have been published every year since the 2001 report, published in 2002. The design of the index in the new Swedish public PAYG pension system indicates that separating the components of the IRR adds new options for designing the indexation of pensions. See the legislative history of the income index and automatic balance mechanism in annex A and Settergren (2001, 2002).

References

- Aaron, H. J. 1966. "The Social Insurance Paradox." *Canadian Journal of Economics* 32 (August): 371–74.
- Arthur, W. B., and G. McNicoll. 1978. "Samuelson, Population and Intergenerational Transfers." *International Economic Review* 10 (1): 241–46.
- Buchanan, J. 1968. "Social Insurance in a Growing Economy: A Proposal for Radical Reform." *National Tax Journal* 21: 386–95.
- Disney, R. 1999. "Notional Accounts as a Pension Reform Strategy: An Evaluation." Social Protection Discussion Paper 9928, World Bank, Washington, DC.
- Federal Old-Age and Survivors Insurance and Disability Insurance Trust Funds, Board of Trustees. 2003. *2003 Annual Report of the Board of Trustees of the Federal Old-Age and Survivors Insurance and Disability Insurance Trust Funds*. Washington, DC: U.S. Government Printing Office.
- Feldstein, M. 2002. "Introduction, an American Perspective." In *Social Security Pension Reform in Europe*, ed. M. Feldstein and H. Siebert, 1–8. National Bureau of Economic Research. Chicago: The University of Chicago Press.
- Iyer, S. 1999. *Actuarial Mathematics of Social Security Pensions*. Geneva: International Labor Organization.
- Keyfitz, N. 1985. *Applied Mathematical Demography*. Berlin and Heidelberg: Springer Verlag.
- . 1988. "The Demographics of Unfunded Pensions." *European Journal of Population* 1: 5–30.
- Lee, R. D. 1980. "Age Structure, Intergenerational Transfers and Economic Growth: An Overview." *Revue Economic* 31 (6): 1129–56.
- . 1988a. "Induced Population Growth and Induced Technological Progress: Their Interaction in the Acceleration Phase." *Mathematical Population Studies* 1 (3): 265–88.
- . 1988b. "Intergenerational Flows of Time and Goods: Consequences of Slowing Population Growth." *Journal of Political Economy* 96 (31): 618–51.

- . 1994a. "The Formal Demography of Aging, Transfers, and the Economic Life Cycle." In *The Demography of Aging*, ed. L. Martin and S. Preston, 8–49. Washington, DC: National Academy Press.
- . 1994b. "Population Age Structure, Intergenerational Transfer, and Wealth: A New Approach, with Application to the U.S." *Journal of Human Resources* 29 (4): 1027–63.
- Lee, R. D., and H. Yamagata. 2003. "Sustainable Social Security: What Would It Cost?" *National Tax Journal* 56 (1, part 1): 27–43.
- Orsag, P., and J. Stiglitz. 1999. "Rethinking Pension Reform: Ten Myths About Social Security Systems." Paper presented at World Bank conference "New Ideas About Old Age Security," Washington, September 14–15.
- Riksförsäkringsverket. 2002. *The Swedish Pension System Annual Report 2001*. Stockholm.
- . 2003. *The Swedish Pension System Annual Report 2002*. Stockholm.
- . 2004. *The Swedish Pension System Annual Report 2003*. Stockholm.
- Samuelson, P. 1958. "An Exact Consumption-Loan Model of Interest With or Without the Social Contrivance of Money." *The Journal of Political Economy* 6 (December): 467–82.
- . 1975a. "The Optimum Growth Rate for Population." *International Economic Review* 16 (3): 531–38.
- . 1975b. "The Optimum Growth Rate for Population: Agreement and Evaluations." *International Economic Review* 17 (2): 516–525.
- Settergren, O. 2001. "The Automatic Balance Mechanism of the Swedish Pension System—A Non-technical Introduction." *Wirtschaftspolitische Blätter* 4: 339–49.
- . 2002. "Financial and Inter-Generational Balance? An Introduction to How the Swedish Pension System Manages Conflicting Ambitions." *Scandinavian Insurance Quarterly* 2: 99–114.
- Settergren, O., and B. D. Mikula. 2001. "Financial Balance & Inter-Generational Fairness in Pay-As-You-Go Pension Systems—Empirical Illustrations in an Alternative Framework." Paper prepared for presentation at the meeting of the Old-age, Invalidity and Survivors' Benefits Technical Commission held during the General Assembly of the International Social Security Association in Stockholm, September.
- Valdés-Prieto, S. 2000. "The Financial Stability of Notional Accounts Pensions." *The Scandinavian Journal of Economics* 102 (3): 395–417.
- Willis, R. J. 1988. "Economics of Changing Age Distribution in Developed Countries." In *Life Cycles, Institutions, and Population Growth: A Theory of the Equilibrium Interest Rate in an Overlapping Generations Model*. *International Studies in Demography*, ed. R. Lee, 106–138. Oxford: Oxford University Press.

Discussion of “The Rate of Return of Pay-As-You-Go Pension Systems: A More Exact Consumption-Loan Model of Interest”

*Ronald Lee**

WHAT RATE OF RETURN IS EARNED BY PARTICIPANTS in a pay-as-you-go (PAYG) pension system? We all know that in steady state, the rate of return equals the population growth rate plus the rate of productivity growth, or equivalently the rate of growth of the tax or contribution base, or the rate of growth of the economy. But many of us casually replace the qualifying condition “steady state” with the condition “mature,” and conclude that the pension programs of most industrial nations should yield this rate of return since most cover the great majority of workers and are therefore mature.

Not so, the authors of this chapter tell us. Mature pension systems depart from the steady state characterization for many reasons, including changes in population age distributions, ages at retirement, ages at beginning work, mortality, rates of productivity growth, and the age structure of wages. Mortality has been declining for more than a century in the industrial nations, and most analysts expect these declines to continue. Even if most other aspects of the pension program were in steady state, this one factor would mean that the effective rate of return earned by each generation in a strict PAYG system would exceed the growth rate of the contribution base. Each generation of workers would pay taxes to cover the retirement costs of generations whose retirements are cut short by mortality at earlier ages than will be true for the generations of workers when they themselves retire several decades later. For this reason, their rate of return will exceed the growth of the contribution base. Of course, to fund this lengthening retirement and higher rate of return, the contribution rate must be rising over time. The effect of declining mortality on the generational rate of return is quite noticeable in the U.S. social security system, adding perhaps 0.3 to 0.5 percent per year to the implicit rate of return, a little realized fact.

The rate of return earned by any particular generation can always be calculated ex post, once its last members have died. But the last members of a generation will not have died until it is well over 100 years old, and by then the question will be of only historical interest. For practical purposes, we may want a measure of the system’s rate of return based on its circumstances and performance today.

The chapter addresses this problem by constructing a measure of the implicit rate of return to a PAYG pension system. The authors first develop a measure for the steady state context, deriving and interpreting an accounting identity. This part of the chapter builds

* Ronald Lee is director of the Center on the Economics and Demography of Aging, University of California at Berkeley.

Research on which this comment was based was supported by a grant from the National Institute of Aging, R37-AG11761.

on an existing literature describing an accounting framework for intergenerational transfers in general. The accounting identity states that the implicit debt of the system, V , or equivalently the present value of the system's current net liabilities, is equal to the annual flow of contributions, C , times the difference between the average age at which a pension payment is received, minus the average at which a contribution (payroll tax) is paid. The authors call this age difference the turnover duration, or TD. The TD synthesizes into a single number a great deal of information about the system's rules, the age distribution of the population, age patterns of labor supply and earnings, and survival. Note that the TD will be greater when pension benefits are indexed to rise with wage growth following retirement than when they are indexed only to inflation, as in the United States. TD would be shorter for a NDC program in which the beneficiary received a lump-sum payment on retirement, and then converted this to an annuity through the private sector. In these and many other ways, the TD will reflect the details of the particular pension plan.

Annex D of the chapter presents rough estimates of the TD for 41 countries, based mainly on their labor force participation rates. These 41 countries, with very different demographic situations, retirement ages, and age profiles of earnings, have remarkably similar TDs, all falling between 31 and 35 years. I find this very surprising, but perhaps the similarity reflects the strong simplifying assumptions used for the calculation. The number given for the United States, 33 years, is right on target, which I know from my own detailed calculations.

We can try out this simple equation for the United States, using the age difference of 33 years. Social security benefit payments (for pensions, survivors, and disability insurance, OASDI) in 2004 amounted to 4.33 percent of GDP, and in a pure PAYG system, this would also be the share of payroll tax payments.¹ The cross-sectional estimate of net pension obligations, or implicit debt for current participants, is therefore 33×4.33 percent of GDP, or 1.43 times GDP. This comes out to US\$16.5 trillion, from which the trust fund of US\$1.5 trillion should be subtracted to get US\$15 trillion. This is considerably greater than the US\$11.2 trillion reported by the actuaries in the Trustees Report (2005: Table IV.B8) as the unfunded obligation for past and current participants. However, the actuaries use a much higher discount rate than the population growth rate which is implicitly used in this cross-sectional calculation and which is probably responsible for most of the discrepancy (see Lee 1994 for a similar estimate for the United States).

The authors suggest that the turnover duration can be interpreted as a measure of the time preference of the pension program, and that its inverse is a measure of the discount rate for contributions. Based on the range of values described above, the implied discount rate would be between 2.8 and 3.2 percent per year. I found this idea puzzling, and would be interested to see the authors develop it further and explain it more clearly.

The results in the first part are interesting and conceptually useful, but they are of limited practical use since actual systems are not in steady state. The results reveal the parameters upon which financial stability is dependent, within the framework of a continuous financial balance sheet. What is unique is that this method of presentation of the framework follows a traditional "business" financial balance, while it introduces the economist's or economic demographer's way of describing this balance. To use this framework for actual, nonsteady state situations, we have to imagine stopping time at two intervals and using a comparative static comparison between them. This is the approach developed by the authors in the second part of the chapter. If we consider a simple special case of their more general equation, in which the trust fund is zero, then the steady state identity can be written $PL = TD \times C$, where PL is pension liability (which I have been calling the implicit debt owed to current participants), TD is the turnover duration just discussed, and C is the annual flow of contri-

butions to the system. They argue that the rate of return on contributions to the system (in a cross-sectional sense) is the rate of return at which the change in PL is zero over time. I would have liked to see more discussion of this point. In a growing economy, fixing the size of the PL would mean that pension wealth would decline relative to GNP, which does not seem right. In any event, setting equal to zero the time derivative of the equation just given, their result implies that the IRR of the system equals the exponential rate of growth of contributions, C , plus the exponential rate of change of the turnover duration, TD (for the special case in which there is no trust fund balance).

In the steady state case, the rate of growth of contributions will just equal the rate of population growth plus the rate of growth of productivity, while the rate of change of TD would be zero. This gives the well-known steady state rate of return for a PAYG system. However, out of steady state, the growth rate of C will also depend on demographic fluctuations rippling through the labor force, changes in the age structure of earnings, changes in retirement age, and changes in the age at starting work, in addition to productivity growth and general population growth.

Similarly, out of steady state, TD will vary due to changes in many of the factors just mentioned, plus changes in survival in old age. For example, if life expectancy is rising so that retirees continue to receive pensions at higher ages, then the average age of receiving pensions will increase and so will TD , and therefore the IRR received through the pension system will rise as well. If life expectancy ceases to rise, then TD will stop rising, and the rate of return will drop back down again. TD will also be changing in other ways due to rising life expectancy, depending on the details of the program. If benefits are indexed to mortality, then younger generations of retirees will receive lower benefit flows, altering TD , for example. To take another example, if the peak of the cross-sectional age-earnings profile shifts to older ages reflecting an increasingly educated labor force, that would raise the age of paying contributions, leading to a reduction in TD and a lower rate of return. All these year-to-year changes are reflected in the suggested calculation, in addition to the standard rate of growth of contributions. The cross-sectional rate of return measure is affected only so long as these changes are occurring. Once the changes stop, the rate of return will also tend to go back to its steady state level, which is unaffected by the level of the TD .

This general approach, and its strengths and hazards, are familiar to demographers who confront similar issues when measuring fertility or mortality. There is a straightforward measure that can be calculated at the end of the reproductive years, completed cohort fertility, and another that can be calculated after all in a generation have died, the average age at death or cohort life expectancy. But for obvious reasons, we would like a measure of fertility and mortality that describes the current situation. Demographers construct synthetic cohort measures, or period measures, that summarize the current situation by imagining that a generation lives its entire life experiencing the age-specific rates of a given year. In the case of fertility, this yields the most commonly used measure, the period total fertility rate (TFR). For mortality it yields another most commonly used measure, period life expectancy. However, there has been intense controversy about these measures, particularly the TFR, because it provides a distorted indication of the completed fertility of any cohort when the timing of fertility in a woman's life cycle is changing—the “tempo” effect—as opposed to the “quantum” of completed cohort fertility. In Europe, the mean age at first birth has been rising by about 0.2 years of age per calendar year for several decades, which depresses the TFR by about 20 percent or nearly 0.4 births relative to the likely eventual completed cohort fertility. Many demographers have suggested adjustments to the period TFR to make it a better indicator of cohort completed fertility.

In the case of the proposed cross-sectional measure of the rate of return to pension participation, similar issues are bound to arise, but have not yet been deeply explored. However, this chapter makes a very promising and enlightening start on addressing this problem in its second part. We can think of the change over time in the turnover duration as introducing a kind of tempo distortion. When the turnover duration is increasing, for example, we would expect that the current rate of benefit payments would be temporarily reduced, and the reverse when the duration is contracting. The rate of change in the contribution rate will likewise be affected by transitory changes in the age of starting or ending work, which come on top of the effects of productivity growth and in a sense distort it. These effects are correct for the cross section, but may be misleading when we try to draw inferences from the cross-sectional rate of return about the longitudinal or long-run rate of return.

In an NDC system, changes across generations in age at retirement and in life expectancy would presumably have no effect on the generational rate of return, since NDC is actuarially fair. But such changes would affect the timing of the payment of contributions and the timing and level of benefits, and these timing changes would be reflected in a changing cross-sectional TD, and thus in the cross-sectional rate of return. The trick is to choose an accounting method, whether cross-sectional or longitudinal, that treats the different generations fairly in a nonsteady state system, avoiding the kinds of inequities that are illustrated clearly in the very helpful annex B.

The chapter is relatively brief, even including its useful appendices, and many details are not spelled out. Discussion toward the end suggests that the cross-sectional pension liability that they have in mind, $PL(t)$, is actually based on an actuarial projection into the future. I found this confusing because there is also a cross-sectional measure of PL , based on the contributions from and benefit payments to the synthetic cohort, and I don't understand why this is not used instead. Similarly, the measure of transfer duration is referred to as "expected" TD toward the end of the chapter, which suggests again a longitudinal and projected aspect, whereas there is a purely cross-sectional measure of TD that does not involve expectations. Because of these details, I do not fully understand the calculations or their underlying logic. Doubtless most of these issues could be resolved through discussion with the authors.

In other places, I had difficulty following because of unfamiliarity with some of the terms and phrases used. This was particularly true of the important discussion toward the end of the "rate of available indexation," "rate of implicit revaluation," "rate of explicit indexation," and "system net income." I am not clear about the relation of these terms and phrases to the IRR, the main topic of this paper. I believe, however, that these terms refer to a kind of residual correction to the cross-sectional estimate of the rate of return.

The method described in this study appears to be a powerful tool for assessing the performance of PAYG pension systems. However, questions remain, and we will need to see a more complete exposition of this approach, and perhaps a more complete analysis, before fully understanding what is being proposed for the rate of return out of steady state. Proper evaluation of the proposed procedure can come only after a more complete understanding of it by the research community. This is quite possibly just a matter of fuller exposition.

In my view, the authors are too modest about their contribution on this topic. While the literature they cite, including my own contributions, is certainly highly relevant, most of it takes a comparative steady state approach and does not deal with nonsteady situations, and none of it addresses the specific problem of how to measure the rate of return to a nonsteady state PAYG pension system. The work in this chapter is an important application

that will, I hope, lead to a series of articles by the authors and by others that further develop these ideas and probe their limitations and uses.

Note

1. See Board of Trustees (2005).

References

- Board of Trustees. 2005. *The 2005 Report of the Board of Trustees of the Federal Old-Age and Survivors Insurance and Disability Insurance Trust Funds*. Washington, DC: United States Government Printing Office.
- Lee, R. 1994. "The Formal Demography of Population Aging, Transfers, and the Economic Life Cycle." In *The Demography of Aging*, ed. L. Martin and S. Preston, 8–49. Washington, DC: National Academy Press.

